# Math 1522 - Final Exam Study Guide

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## Contents

# Summary and Disclaimer

This is a study guide for the final exam for math 1522 at the University of New Mexico (Calculus II). The exam covers chapters 6, 7, 11, and sections 9.1 through 9.3 of Stewart's Calculus, as well as basic complex numbers (which can be found in Appendix H of the same text). As such, this study guide is focused on that material. I assume that the student reading this study guide is familiar with the material from a calculus 1 course and the material previously covered in calculus 2. If a you feel that you need to review this material, you can send me an email, or take a look at Paul's Online Math notes:

https://tutorial.math.lamar.edu/

As the final exam for Calculus 2 at the University of New Mexico is a common exam, everybody taking this course at the University of New Mexico should be able to get some use out of this study guide.

# Methods and Techniques

The primary focus of Calculus 2 is on integration and integration techniques. However, we first had to discuss several "bits and bobs" so to speak. This is chapter 6 of Stewart's Calculus.

The most important result that we have from this section is the inverse function theorem:

**Inverse Function Theorem**

If $f$ is a differentiable one-to-one function and $f(a) = b$, then

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

It is also good to know the derivatives of exponential functions and logarithms:

**Derivatives of Exponential Functions and Logarithms**

To find the derivatives of exponential functions, we only need to remember the two rules

$$(\log_b(x))' = \frac{1}{x\ln(b)} \qquad \text{and} \qquad (b^x)' = \ln(b)b^x.$$

If $b = e$, this gives us the special cases

$$(\ln(x))' = \frac{1}{x} \qquad \text{and} \qquad (e^x)' = e^x.$$

Finally, from the first section of the course, we should know L'Hopital's rule.

**L'Hopital's Rule**

To evaluate a limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

Where the limit is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, we can find the derivatives. In other words:

$$\lim x \to a \frac{f(x)}{g(x)} = \lim x \to a \frac{f'(x)}{g'(x)}$$

After the above material, our focus moves toward integration. For this, you should know integration by parts.

**Integration by Parts**

Since integrals undo derivatives, we want to find out what happens when we undo the chain rule. This gives us integration by parts, which has the following form.

$$\int u \; dv = uv - \int v \; du.$$

To do this in practice, you pick $u$ to be the thing in the integral that you don't know how to integrate, but that you do know how to differentiate, and you let $dv$ be everything else.

It is also good to know the following trigonometric identities:

**Useful Trigonometric Identities**

First, we have the Pythagorean identity. It is called this because it comes from the definition of sin and cos combined with the Pythagorean Theorem.

$$\sin^2(x) + \cos^2(x) = 1$$

Next, we have two formulas which come from rearranging the double angle identities. They are called this because they tell us how $\cos(2x)$ relates to $\sin^2(x)$ and $\cos^2(x)$.

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

These are useful for evaluating trigonometric integrals.

**Trigonometric Integrals**

We wish to evaluate integrals of the form

$$\int \sin^n(x)\cos^m(x)\ dx.$$

If $n$ is odd, we can write this as

$$\int (1-\cos^2(x))^{\frac{n-1}{2}}\cos^m(x)\sin(x)\ dx$$

and substitute in $u=\cos(x)$ to solve. If $m$ is odd, we can write the integral as

$$\int \sin^n(x)(1-\sin^2(x))^{\frac{m-1}{2}}\cos(x)\ dx$$

and set $u=\sin(x)$ to solve.
If $m$ and $n$ are both even, we use the double angle identities to change them both into terms of $\cos(2x)$, and solve that integral.

This allows us to do trigonometric substitutions:

**Trigonometric Substitution**

We would like to solve integrals containing an $x^2+a^2$ term, $x^2-a^2$ term, or $a^2-x^2$ term, where $a$ is some constant number (think, 1, 2, or even $\sqrt{3}$). We do the following substitutions, and solve the resulting integral:

| Equation | Substitution |
|---|---|
| $x^2+a^2$ | $x=a\tan(\theta)$ |
| $x^2-a^2$ | $x=a\sec(\theta)$ |
| $a^2-x^2$ | $x=a\sin(\theta)$ |

Each of these substitutions has an associated triangle which allows us to switch out of terms of $\theta$ at the end of our integration.

At this point, our course redirected itself toward series, with the goal of Taylor series. To start this, we discussed geometric series, which often come up in a discussion of series. This is followed by telescoping series, since geometric and telescoping series are often the only kinds of series where computing the value is (somewhat) easy.

**Geometric Series Formula**

The if a geometric series has $|r|<1$, then it has the following value:

$$\sum_{n=k}^{\infty} a_n = \frac{a_k}{1-r}$$

where $r$ is the ratio of the series.

And after this we discussed telescoping series:

**Telescoping Sums**

These are sums of the form

$$\sum_{n=1}^{\infty} f(n) - f(n+k)$$

where $k$ is some positive number. The way to solve these is to write out the actual values of the partial sums

$$s_N = f(1) - f(1+k) + f(2) - f(2+k) + \ldots + f(N) - f(N+k).$$

And then evaluate the limit

$$\lim_{N\to\infty} s_N.$$

And then we have our standard battery of tests:

**Divergence Test**

Consider the series

$$\sum_{n=1}^{\infty} a_n.$$

If

$$\lim_{n\to\infty} a_n \neq 0,$$

then the series must diverge. If the limit goes to zero, then the test tells you nothing about the series.

**Integral Test**

If $f(x)$ is continuous and eventually decreasing, then

$$\sum_{n=k}^{\infty} f(n)$$

converges if the integral

$$\int_k^\infty f(x)\ dx$$

converges, and diverges otherwise.

**$p$-test**

The series

$$\sum_{n=k}^\infty \frac{1}{n^p}$$

converges if $p > 1$ and diverges otherwise.

**Limit Comparison Test**

Consider $a_n$ and $b_n$ as two non-negative sequences. Then if

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

is a positive (non-zero) finite number, then

$$\sum_{n=k}^\infty a_n$$

and

$$\sum_{n=k}^\infty b_n$$

either both converge or both diverge.

**Ratio Test**

Let $a_n$ be given. If

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

is defined, then if $L > 1$, the series diverges, if $L < 1$ the series converges absolutely, and if $L = 1$ the test is inconclusive.

**Alternating Series Test**

If $\lim\limits_{n\to\infty} b_n = 0$ and $b_n$ is a decreasing sequence, then $\displaystyle\sum_{n=k}^\infty (-1)^n b_n$ converges.

Which leads into absolute convergence:

**Absolute Convergence**

The series

$$\sum_{n=k}^{\infty} a_n$$

is said to converge absolutely if

$$\sum_{n=k}^{\infty} |a_n|$$

converges.

And the error theorem on alternating series (which we will get to later when we cover Taylor's Estimation Theorem).

Taylor series themselves are given by

**Taylor Series**

The Taylor series for a function $f(x)$ centered at $x = a$ has the formula

$$\sum_{n=0}^{\infty} f^{(n)}(a)\frac{(x-a)^n}{n!}$$

And the error term on alternating series is given by

**Error on Alternating Series**

The difference between the $n$th partial sum and the actual value of

$$\sum_{n=1}^{\infty} (-1)^n b_n$$

is less than $b_{n+1}$.

On a similar note, the error term on general Taylor series is given by

**Taylor's Estimation Theorem**

The difference between $p_n(x)$ (centered at $a$) and $f(x)$ on an interval $[b, c]$ (equivalently for $b \leq x \leq c$) is given by

$$|p_n(x) - f(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}$$

Where $M$ is the maximum value of $|f^{(n+1)}(x)|$ on the interval $[b, c]$. It is often useful to note that $|x - a|$ is less than the distance from $a$ to the furthest endpoint of the interval (be it $b$ or $c$).

It is also good to know Euler's Formula, just in case it crops up:

**Euler's Formula**

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Finally, we get to ordinary differential equations and initial value problems.

**Separable Differential Equations**

If we are given a differential equation

$$\frac{dy}{dx} = F(x, y)$$

and we can separate it as

$$\frac{dy}{dx} = f(x)g(y),$$

then we have that

$$\int \frac{dy}{g(y)} = \int f(x)\ dx.$$

And lastly we have integrating factors.

**Integrating Factors**

If we have a function of the form

$$y' + f(x)y = g(x),$$

we can solve it using an integrating factor. Here, I will call the integrating factor $I(x)$, although some prefer to call it $\mu(x)$. We first multiply by it to get

$$I(x)y' + I(x)f(x)y = I(x)g(x).$$

Then, we know that $(I(x)y) = y'I(x) + I'(x)y$ by the chain rule. So, we want to find $I(x)$ so that

$$I(x)y' + I(x)f(x)y = I(x)y'(x) + I'(x)y.$$

Canceling several terms gives us that

$$I'(x) = I(x)f(x).$$

Integrating this gives you the value of $I$ such that

$$(I(x)y)' = I(x)g(x).$$

Integrating this then solves the differential equation for the general solution.

# Worked Examples

For a set of worked examples, please see the other exam review guides, as well as my recitation notes.

# Practice Problems

For a list of practice problems, please see the other exam review guides, as well as my recitation notes.

## Unsolved Questions

As I am no longer allowed to distribute practice exams, here is a list of unsolved questions which I believe are of similar difficulty to what might be asked of you on an exam.

1. Evaluate the integral
$$\int \frac{1+x-x^2}{x^2}$$

2. Evaluate the integral
$$\int_0^\infty x^2\ dx$$

3. Evaluate the integral
$$\int \tanh(x)\mathrm{sech}^2(x)\ dx$$

4. Evaluate the integral
$$\int_5^\infty xe^{-x}\ dx$$

5. If $\displaystyle\sum_{n=0}^\infty a_n = 3$, what are the values of
$$\lim_{n\to\infty} a_n \qquad \text{and} \qquad \lim_{n\to\infty} s_n$$

6. Determine if the series diverges, converges conditionally, or converges absolutely. Determine its value if possible.
$$\sum_{n=0}^\infty (-1)^n \frac{3^n}{n!}$$

7. Determine if the series diverges, converges conditionally, or converges absolutely. Determine its value if possible.
$$\sum_{n=1}^\infty (-1)^n \frac{(n+1)(n-1)}{n^3-n}$$

8. Determine if the series diverges, converges conditionally, or converges absolutely. Determine its value if possible.
$$\sum_{n=0}^\infty (-1)^n \frac{2\cdot 3^{2n+1}}{5^n}$$

9. Find the power series for the following function, and state its interval of convergence:
$$f(x) = \sin(-2x^2)$$

10. Find the power series for the following function, and state its interval of convergence:

$$f(x) = -\frac{x}{(1-x)^2}$$

11. Find the interval of convergence of the following power series.

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

12. Find the interval of convergence of the following power series.

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{3^n}$$

13. Write $z = -2 - \sqrt{3}i$ in polar form.

14. Write $z = -1 + i$ in polar form.

15. Write $z = \sqrt{3}e^{\frac{2\pi}{3}i}$ in standard form.

16. Solve the initial value problem

$$2y' \cos(x) = \frac{\sec(x)}{y}, \qquad y(0) = -1.$$

17. Solve the initial value problem

$$y' = 2x + y \qquad y(0) = -1.$$

18. Solve the initial value problem

$$e^x y' = e^{2x} y + 1 \qquad y(0) = 0.$$

19. Solve the initial value problem

$$y' = e^{x^2 - y} x, \qquad y(0) = 1.$$

20. Solve the initial value problem

$$y' = 2x^2 y \qquad y(1) = 1.$$